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UPPER-THRESHOLDS FOR SHOCK FORMATION IN TWO-DIMENSIONAL WEAKLY RESTRICTED EULER–POISSON EQUATIONS*

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Abstract. The multi-dimensional Euler–Poisson system describes the dynamic behavior of many important physical flows, yet as a hyperbolic system its solution can blow up for some initial configurations. This paper strives to advance our understanding on the critical threshold phenomena through the study of a two-dimensional weakly restricted Euler–Poisson (WREP) system. This system can be viewed as an improved model of the restricted Euler–Poisson (REP) system introduced in [H. Liu and E. Tadmor, Comm. Math. Phys., 228:435–466, 2002]. We identify upper-thresholds for finite time blow up of solutions for WREP equations with attractive/repulsive forcing. It is shown that the thresholds depend on the size of the initial density relative to the initial velocity gradient through both trace and a nonlinear quantity.

Keywords. critical thresholds, restricted Euler–Poisson equations

AMS subject classifications. Primary 35Q35; Secondary 35B30

1. Introduction and statement of main results

We are concerned with the threshold phenomenon in two dimensional Euler–Poisson equations. The pressure-free Euler–Poisson (EP) equations in multiple dimensions are

$$\rho_t + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1.1a}$$

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = k\nabla\Delta^{-1}(\rho - c_b), \tag{1.1b}$$

which are the usual statements of the conservation of mass and Newton's second law. Here k is a physical constant which parameterizes the repulsive ($k>0$) or attractive ($k<0$) forcing. Also, c_b denotes the constant background state. The local density $\rho = \rho(t,x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^+$ and the velocity field $\mathbf{u}(t,x) : \mathbb{R}^+ \times \mathbb{R}^2 \mapsto \mathbb{R}^2$ are the unknowns. This hyperbolic system (1.1) with non-local forcing describes the dynamic behavior of many important physical flows, including semi-conductors and plasma physics ($k>0$), and the collapse of stars due to self gravitation ($k<0$) [1,2,6,10,20,22].

There is a considerable amount of literature relevant to the behavior of solutions to the Euler–Poisson equations. Let us mention the study of steady-state solutions [8,20] and the global existence of weak solutions [21]. Global existence due to damping relaxation and with non-zero background can be found in [23]. Construction of a global smooth irrotational solution in three and two dimensional space can be found in [9,11].

To address the fundamental question of the persistence of C^1-regularity for solutions of the Euler–Poisson system and related models, the concept of *critical threshold* (CT) was originated and developed in a series of papers by Engelberg, Liu and Tadmor [7,16–19] and more. The critical threshold in [7] describes the conditional stability of the 1D Euler–Poisson system, where the answer to the question of global vs. local existence depends on whether the initial data crosses a critical threshold. Following [7], critical thresholds have been identified for several one dimensional models, including 2×2 quasi-linear hyperbolic relaxation systems [15], Euler equations with non-local interaction and alignment forces [3], and traffic flow models [14].

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Moving to the multi-dimensional Euler-Poissiion system, one has to identify the proper quantities to describe the critical threshold phenomenon. Liu and Tadmor introduce in [16] the method of spectral dynamics which relies on the dynamical system governing eigenvalues of the velocity gradient matrix, $M := \nabla \mathbf{u}$, along particle paths.

We follow their approach and, in order to trace the evolution of M, we take gradient of Equation (1.1b) to find

$$\partial_t M + \mathbf{u} \cdot \nabla M + M^2 = k \nabla \otimes \nabla \Delta^{-1} [\rho - c_b] = k R[\rho - c_b] \tag{1.2}$$

where R is the Riesz matrix operator, defined as

$$R[f] := \nabla \otimes \nabla \Delta^{-1} [f] = \mathcal{F}^{-1} \left\{ \frac{\xi_j \xi_k}{|\xi|^2} \hat{f}(\xi) \right\}_{j,k=1,2}.$$

Now, the Euler–Poisson equations are recast into the coupled system

$$\begin{aligned} &\frac{d}{dt} M + M^2 = k R[\rho - c_b], \\ &\frac{d}{dt} \rho + \rho \operatorname{tr} M = 0, \end{aligned} \tag{1.3}$$

with $\frac{d}{dt}$ standing for the usual convective derivative, $\partial_t + \mathbf{u} \cdot \nabla$. It is the non-local forcing, $kR[\rho - c_b]$, which presents the main obstacle to studying the critical threshold phenomenon of the multi-dimensional Euler–Poisson equations.

To gain a better understanding of the dynamics of velocity gradient M governed by System (1.3), Liu and Tadmor introduce in [16] the restricted Euler–Poisson (REP) system (1.4), which is obtained from System (1.3) by restricting attention to the local isotropic trace $\frac{k}{n}(\rho - c_b) I_{n \times n}$ of the global coupling term $kR[\rho - c_b]$, namely,

$$\begin{aligned} &\frac{d}{dt} M + M^2 = \frac{k}{n} (\rho - c_b) I_{n \times n}, \\ &\frac{d}{dt} \rho + \rho \operatorname{tr} M = 0. \end{aligned} \tag{1.4}$$

Replacing the nonlocal forcing term by a local one, it was shown that in the repulsive case, the restricted $2D$ REP model admits a two-sided critical threshold [17]. For arbitrary $n \geq 3$ dimensional REP model, the author and Liu identified both upper-thresholds for finite time blow up of solutions and sub-thresholds for global existence of solutions [13].

In this work, we propose the weakly restricted Euler–Poisson (WREP) system as a *semi-localized* alternative to Equation (1.3). Specifically, we consider a gradient flow M, governed by

$$\begin{aligned} &\frac{d}{dt} M + M^2 = \begin{pmatrix} \frac{k}{2}(\rho - c_b) & R_{12} \\ R_{21} & \frac{k}{2}(\rho - c_b) \end{pmatrix}, \\ &\frac{d}{dt} \rho + \rho \operatorname{tr} M = 0, \end{aligned} \tag{1.5}$$

subject to initial data

$$(M, \rho)(0, \cdot) = (M_0, \rho_0).$$

We then investigate threshold conditions on the initial data that lead to the finite time blow-up of M.

To state our main results, we introduce several quantities with which we characterize the behavior of the velocity gradient tensor M. The first one is the trace $d := \mathrm{tr}M = \nabla\cdot\mathbf{u}$, and the second one is a nonlinear quantity defined by

$$B = \eta^2 - \omega^2, \tag{1.6}$$

where $\omega := M_{21} - M_{12}$ and $\eta := M_{11} - M_{22}$. We use the notation $B_0 = \eta_0^2 - \omega_0^2$ and $d_0 = \mathrm{tr}(M_0)$.

Theorem 1.1 (Repulsive forcing, k >0). *Consider the 2-dimensional, repulsive weakly-restricted Euler–Poisson system* (1.5). *Then, the solution to the 2D WREP system will blow up in finite time if the initial data* (ρ_0, M_0) *lies in one of the following three regions,* $(\rho_0, d_0, B_0) \in S_1 \cup S_2 \cup S_3$*:*

(i) $(\rho_0, d_0, B_0) \in S_1$,

$$S_1 := \left\{ (\rho, d, B) \middle| \rho > 0, \quad B > \frac{k\rho^2}{2c_b} \right\}.$$

(ii) $(\rho_0, d_0, B_0) \in S_2$,

$$S_2 := \left\{ (\rho, d, B) \middle| \rho > 0, \quad B = \frac{k\rho^2}{2c_b} \quad \text{and} \quad (\rho, d) \neq (2c_b, 0) \right\}.$$

(iii) $(\rho_0, d_0, B_0) \in S_3$

$$S_3 := \left\{ (\rho, d, B) \middle| \rho > 0,\ 0 < B < \frac{k\rho^2}{2c_b} \text{ and either } G(\rho, B) \le 0 \text{ or } d < \mathrm{sign}(F)\sqrt{\rho G(\rho, B)} \right\},$$

where

$$F = \rho - 2c_b - \sqrt{\rho^2 - 2c_b B/k},$$
$$G = \frac{B - 2kc_b}{\rho} - 2\sqrt{k^2 - \frac{2kc_b B}{\rho^2}} - 2k\log\left(\frac{\rho - \sqrt{\rho^2 - 2c_b B/k}}{2c_b} \right).$$

Theorem 1.2 (Attractive forcing, k <0). *Consider the 2-dimensional, attractive weakly-restricted Euler–Poisson system* (1.5). *Then the solution to the 2D WREP system will blow up in finite time if the initial data* (ρ_0, M_0) *lies in the following region,* $(\rho_0, d_0, B_0) \in S$*:*

$$S := \left\{ (\rho, d, B) \middle| \rho > 0, \quad B > 0 \quad \text{and} \quad d < \mathrm{sign}(F)\sqrt{\rho G(\rho, B)} \right\},$$

where F and G are same as in Theorem 1.2.

The following lemma is crucial in our proofs of main theorems.

Lemma 1.1. *From the 2D WREP system*

$$\frac{d}{dt}M + M^2 = \begin{pmatrix} \frac{k}{2}(\rho - c_b) & R_{12} \\ R_{21} & \frac{k}{2}(\rho - c_b) \end{pmatrix}, \quad \frac{d}{dt}\rho + \rho\,\mathrm{tr}M = 0,$$

we can derive the following system of closed ordinary differential inequalities (ODI):

$$\begin{aligned} d' &\le -\frac{1}{2}d^2 + \frac{1}{2}\left\{ \left(\frac{\omega_0}{\rho_0}\right)^2 - \left(\frac{\eta_0}{\rho_0}\right)^2 \right\}\rho^2 + k(\rho - c_b), \qquad ' := \frac{d}{dt} \\ \rho' &= -d\rho. \end{aligned} \tag{1.7}$$

Proof. From the matrix equation (1.5), namely,

$$\frac{d}{dt}M+\begin{pmatrix} M_{11}^2+M_{12}M_{21} & dM_{12} \\ dM_{21} & M_{12}M_{21}+M_{22}^2 \end{pmatrix}=\begin{pmatrix} \frac{k}{2}(\rho-c_b) & R_{12} \\ R_{21} & \frac{k}{2}(\rho-c_b) \end{pmatrix},$$

we obtain

$$\begin{cases} \eta'+\eta d=0, \\ \omega'+\omega d=R_{12}-R_{21}=0, \\ \rho'+\rho d=0. \end{cases} \tag{1.8}$$

Hence

$$\frac{\eta}{\eta_0}=\frac{\omega}{\omega_0}=\frac{\rho}{\rho_0}.$$

Thus,

$$\begin{aligned} d' &= -(M_{11}^2+M_{22}^2)-2M_{12}M_{21}+k(\rho-c_b) \\ &= -\left\{M_{11}^2+M_{22}^2+\frac{(M_{12}+M_{21})^2}{2}\right\}+\frac{(M_{12}-M_{21})^2}{2}+k(\rho-c_b) \\ &\le -\frac{(M_{11}+M_{22})^2+(M_{11}-M_{22})^2}{2}+\frac{1}{2}\omega^2+k(\rho-c_b) \\ &= -\frac{1}{2}d^2-\frac{1}{2}\eta^2+\frac{1}{2}\omega^2+k(\rho-c_b) \\ &= -\frac{1}{2}d^2+\frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2-\left(\frac{\eta_0}{\rho_0}\right)^2\right\}\rho^2+k(\rho-c_b). \end{aligned} \tag{1.9}$$

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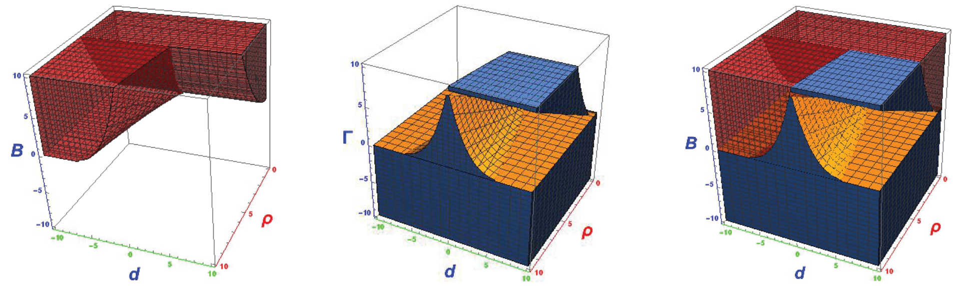
Several remarks are in order.

(1) We compare the current blow-up results with those in [17]. The critical threshold results in the REP (restricted Euler–Poisson system) were formulated in [17] in terms of the spectral gap. That is, in [17] the REP model's sub-critical data is expressed in terms of (ρ_0,d_0,Γ_0). Here, Γ_0 is the initial spectral gap, i.e., $\Gamma_0=(\lambda_2(0)-\lambda_1(0))^2$. In Theorem 1.1, the WREP model's super-threshold data (blow-up condition) is expressed in terms of (ρ_0,d_0,B_0), where $B_0=(M_{11}(0)-M_{22}(0))^2-(M_{21}(0)-M_{12}(0))^2=:\eta_0^2-\omega_0^2$. One can easily derive that $\Gamma_0=B_0+(M_{21}(0)+M_{12}(0))^2$.

For comparison purpose, the REP system's d-dynamics equation in [16] can be re-written as follows:

$$d'=-\frac{1}{2}d^2+\frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2-\left(\frac{\eta_0}{\rho_0}\right)^2-(M_{12}+M_{21})^2\right\}\rho^2+k(\rho-c).$$

One can observe that the coefficient of ρ^2 in the above equation is more negative relative to that in System (1.7) due to the presence of the $-(M_{12}+M_{21})^2$ term. Therefore, we can expect that the blow-up initial configuration set (M_0,ρ_0) of WREP is contained in that of REP. More precisely, in Figure 1.1 (a) and (b), the blow-up initial configuration for WREP and the global existence configuration for REP are plotted in $[-10,10]^3$ cubes with $k=c=0.5$. In the last figure, it is observed that when $(M_{21}(0)+M_{12}(0))^2=0$



(a) *WREP blow-up initial data* (b) *REP global existence initial data in [17]* (c) *Union of WREP blow-up set and REP global existence set*

Fig. 1.1: *Blow-up initial configuration of WREP and global existence initial configuration of REP*

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(which in turn implies $\Gamma_0 = B_0$), the WREP blow-up set and the REP global existence set fit together nicely without any intersection.

(2) Chae, Cheng and Tadmor [4, 5] obtain the blow-up result for n-dimensional *full* Euler–Poisson systems (1.3) with attractive forcing $k<0$. For proofs of the results in [4, 5], the vanishing initial vorticity condition (which ensures $\nabla\times\mathbf{u}=\mathbf{0}$ for all time) is essential to ensure the following Ricatti-type key inequality

$$d' \leq -\frac{1}{n}d^2 + k(\rho - c_b). \tag{1.10}$$

Our Lemma 1.1 tells us that one can derive the similar Ricatti-type inequality (1.7) when the initial vorticity condition $\nabla\times\mathbf{u}_0(x)\neq 0$.

(3) The critical threshold in the 1D Euler–Poisson system depends only on the relative size of the initial velocity slope and the initial density [7]. In contrast to the 1D EP system, the threshold conditions in 2D REP depend on several initial quantities: density ρ_0, divergence $\nabla\cdot\mathbf{u}_0$, vorticity $\nabla\times\mathbf{u}_0$, and gap $v_{0x}-u_{0y}$.

(4) The above results show that to ensure the finite time blow up, a relatively small absolute value of initial vorticity $|u_{0y}-v_{0x}|$ is needed. This fact is consistent with the results in [17] that show that the global smooth solution is ensured if the initial velocity gradient has complex eigenvalues, which applies, for example, to a class of initial configurations with sufficiently large vorticity.

(5) With relatively small initial vorticity, the finite time blow-up occurs if the initial divergence is below a threshold expressed in terms of the initial density and entries in $\nabla\mathbf{u}_0$. So we can view our results in the perspective of the critical thresholds.

We shall conclude this section by discussing some questions for future study. For the question of extension of blow-up results in WREP, from System (1.3), consider the η ODE

$$\eta' + \eta d = k(R_{11} - R_{22}),$$

where R is the Riesz matrix operator. We have no clear idea on how fast η is changing in time because we lack an L^∞ bound on R_{ij}. The key assumption in our WREP model is

$$R_{11}=R_{22}=\frac{k}{2}(\rho-c_b).$$

With this assumption the above η ODE is reduced to a simple ODE, $\eta'+\eta d=0$ in System (1.8) and the ODE allows us to write η in terms of ρ. To extend the blow-up result, an advanced sharp technique with the aid of harmonic analysis seems to be needed. Another effort to advance our understanding of the *full* Euler–Poisson equations may be made by modifying the Riesz transform. In [12], the author propose a *modified* Euler–Poisson equations with a *modified* Riesz transform where the singularity at the origin is removed.

Concerning the question of possible regularity results for sub-critical data, we first notice that the derivation of Ricatti-type dynamics in Lemma 1.1 relies on the assumption $R_{11}=R_{22}$. The one sided Ricatti-type ODI(ordinary differential inequality) in System (1.7) allow us to derive finite time blow-up of d (i.e., $d\to-\infty$) *only*. That is, due to the nature of the one sided inequality, there is no way to bound d from below.

In order to derive a regularity result for sub-critical data, one may need to bound $|d|$ and ρ for all time. This may be possible in the following two cases.

i) One finds a closed system of ODE, not ODI. For example, in [17], Liu-Tadmor considered the 2D REP (restricted Euler-Poisson equation) and derived the following ODE system

$$d'+\frac{d^2+\beta\rho^2}{2}=k\rho, \quad \rho'=-\rho d.$$

Then they studied the dynamics of (ρ,d) parametrized by β (where β is the ratio of the initial spectral gap and the initial density). This ODE system enables one to give a complete description of the critical threshold phenomenon for the 2D REP.

ii) One finds a two sided differential inequality for ρ or d. For example, in [13], the author and Liu considered the n-dimensional REP and derived the following two sided ODI:

$$-n\rho\lambda_n\le\rho'\le-n\rho\lambda_1.$$

(here λ_i is the eigenvalue of $\nabla\mathbf{u}$). This two sided differential inequality led to the desired thresholds for both global existence and finite time blow-up.

With this weakly restricted Euler–Poisson equation, the author was unable to obtain any two sided ODI or closed ODE system which may bound $|d|$ or ρ. Therefore, a regularity result for sub-critical data for WREP is still yet to be found.

2. Proof of Theorem 1.1

In this section we show the existence of an upper threshold for the 2D WREP with $k>0$. Let $A:=\frac{B_0}{2\rho_0^2}=-\frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2-\left(\frac{\eta_0}{\rho_0}\right)^2\right\}$, then the ODI system (1.7) is rewritten as

$$d'\le-\frac{1}{2}d^2-A\rho^2+k(\rho-c_b), \tag{2.1a}$$

$$\rho'=-d\rho. \tag{2.1b}$$

2.1. Cases (i) and (ii): We write (2.1a) as

$$d' \le -\frac{1}{2}d^2 - A\left(\rho - \frac{k}{2A}\right)^2 + \frac{k^2}{4A} - kc_b. \tag{2.2}$$

If $B_0 > \frac{k\rho_0^2}{2c_b}$ (i.e., $k < 4Ac_b$), we have $A > 0$ so the second term in the right hand side of Inequality (2.2) is non-positive. Therefore, it follows that

$$d' \le -\frac{1}{2}d^2 + \frac{k^2}{4A} - kc_b = -\frac{1}{2}d^2 + k\left(\frac{k}{4A} - c_b\right).$$

We see that $k(\frac{k}{4A} - c_b) < 0$, therefore the above inequality ensures that d blows up in finite time for any choice of d_0.

Also, if $B_0 = \frac{k\rho_0^2}{2c_b}$, i.e., $k = 4Ac_b$, then Inequality (2.2) is reduced to

$$d' \le -\frac{1}{2}d^2 - A(\rho - 2c_b)^2.$$

Thus, d blows up in finite time for all initial data except $(\rho_0, d_0) = (2c_b, 0)$.

2.2. Case (iii): We assume that $0 < B_0 < \frac{k\rho_0^2}{2c_b}$ (i.e., $0 < A < \frac{k}{4c_b}$).

For simplicity we set $c_b = 1$ and compare the ODI (2.1) with the following ODE system by still using the original variables.

$$\begin{aligned} d' &= -\frac{1}{2}d^2 - A\rho^2 + k(\rho - 1), \\ \rho' &= -d\rho. \end{aligned} \tag{2.3}$$

Here we note that $k > 4Ac_b = 4A\cdot 1$ because of the assumption in this subsection. Also note that the ODE system admits two distinct critical points, a saddle at $(\rho, d) = (\alpha_1, 0)$, and a spiral at $(\rho, d) = (\alpha_2, 0)$, where

$$(\alpha_1, 0) := \left(\frac{k + \sqrt{k^2 - 4kA}}{2A}, 0\right), \quad (\alpha_2, 0) := \left(\frac{k - \sqrt{k^2 - 4kA}}{2A}, 0\right).$$

We shall use the above facts to construct the threshold via the phase plane analysis. Following the same q-transformation as that employed in [17], we set $q := d^2$ and differentiate along the particle path $\{(t, X(t,a)) \mid X_t(t,a) = u(t, X(t,a)),\ X(t=0,a) = a\}$ to get

$$\frac{dq}{d\rho} = 2d\frac{d'}{\rho'} = \frac{q}{\rho} + 2A\rho - 2k(1 - \frac{1}{\rho}),$$

which yields

$$\frac{d}{d\rho}\left(\frac{q}{\rho}\right) = 2A - 2k\left(\frac{1}{\rho} - \frac{1}{\rho^2}\right). \tag{2.4}$$

Integration leads to a global invariant,

$$\frac{d^2}{\rho} - \frac{d_*^2}{\rho_*} = -2\int_{\rho_*}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\, ds, \tag{2.5}$$

where d_* and ρ_* are some constants. By setting $(\rho_*, d_*) = (\alpha_1, 0)$, we find the separatrix curve passing through $(\alpha_1, 0)$,

$$\frac{d^2}{\rho} = -2\int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\,ds. \tag{2.6}$$

The above curve has two x intercepts. One is at $(\alpha_1, 0)$ and the other is denoted by $(\alpha_3, 0)$ with $0 < \alpha_3 < \alpha_2$. In fact, consider

$$\int_{\rho}^{\alpha_1} \frac{-As^2 + k(s-1)}{s^2}\,ds = \int_{\rho}^{\alpha_2} \frac{-As^2 + k(s-1)}{s^2}\,ds + \int_{\alpha_2}^{\alpha_1} \frac{-As^2 + k(s-1)}{s^2}\,ds.$$

Note that $-As^2 + k(s-1) \geq 0$ for all $s \in [\alpha_2, \alpha_1]$, and $\lim_{\rho\to 0+} \int_{\rho}^{\alpha_2} \frac{-As^2+k(s-1)}{s^2}\,ds \to -\infty$. This proves the existence of the intercept $(\alpha_3, 0)$, and the following identity holds,

$$\int_{\alpha_3}^{\alpha_1} \frac{-As^2 + k(s-1)}{s^2}\,ds = 0. \tag{2.7}$$

Back to ODI system (2.1), the same q-transformation gives us

$$\frac{d}{d\rho}\left(\frac{q}{\rho}\right) \geq 2A - 2k\left(\frac{1}{\rho} - \frac{1}{\rho^2}\right). \tag{2.8}$$

We now discuss subcases distinguished by the location of initial points; see Figure 2.1.

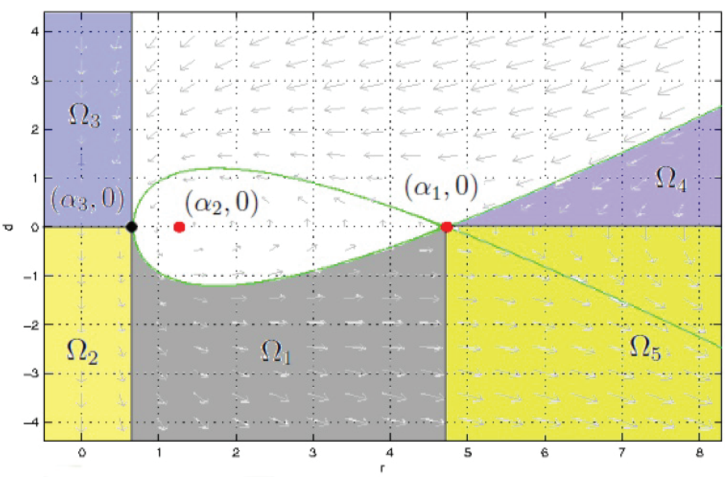

Fig. 2.1: *The blow up region of* $k > 4Ac_b$ *case.*

- $(\rho_0, d_0) \in \Omega_1 := \left\{(\rho, d)\,\middle|\, \alpha_3 \leq \rho \leq \alpha_1,\ d < -\sqrt{2\rho \int_{\rho}^{\alpha_1} \frac{-As^2+k(s-1)}{s^2}\,ds}\right\}$.

  First, we show no orbit of the ODI touches the lower left arc of the separatirx curve (2.6): Inequality (2.8) leads to

$$\frac{d^2}{\rho} - \frac{d_0^2}{\rho_0} \geq -2\int_{\rho_0}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\,ds. \tag{2.9}$$

Now, consider a point (ρ_0, d_*) on the separatrix curve above (ρ_0, d_0), i.e.,

$$\frac{d_*^2}{\rho_0} = -2\int_{\alpha_1}^{\rho_0} \frac{-As^2+k(s-1)}{s^2}\,ds.$$

Since $d_0 < d_* < 0$ we have

$$\begin{aligned}
\frac{d^2}{\rho} &\ge \frac{d_0^2}{\rho_0} - 2\int_{\rho_0}^{\rho} \frac{-As^2+k(s-1)}{s^2}\,ds \\
&> \frac{d_*^2}{\rho_0} - 2\int_{\rho_0}^{\rho} \frac{-As^2+k(s-1)}{s^2}\,ds \\
&= -2\int_{\alpha_1}^{\rho} \frac{-As^2+k(s-1)}{s^2}\,ds.
\end{aligned}$$

This relation shows that if $(\rho_0, d_0) \in \Omega_1$, then no point $(\rho(t), d(t))$ crosses the separatrix curve (2.6).

Next, we show that if $(\rho_0, d_0) \in \Omega_1$, then $(\rho(t), d(t)) \nrightarrow (\alpha_1, 0)$ as $t \to \infty$. Suppose instead that $\rho(t) \nearrow \alpha_1$ and $d(t) \nearrow 0$ as $t \to \infty$, then Inequality (2.9) would imply

$$-\frac{d_0^2}{\rho_0} \ge -2\int_{\rho_0}^{\alpha_1} \frac{-As^2+k(s-1)}{s^2}\,ds.$$

But this contradicts the fact that $(\rho_0, d_0) \in \Omega_1$.

Finally, we show that if $(\rho_0, d_0) \in \Omega_1$, then there exists $t^* < \infty$ such that $\rho(t^*) > \alpha_1$. Suppose we had $\alpha_3 \le \rho(t) \le \alpha_1$, for all $t > 0$, then since $d(t) < 0$ for all time, we would see from $\rho(t) = \rho_0 \exp(-\int_0^t d(s)ds)$ that $d(t)$ must go to 0. Since no orbit can touch the lower left arc, we are left with only one possibility $(\rho(t), d(t)) \to (\alpha_1, 0)$. But we already showed that no orbit from Ω_1 goes to $(\alpha_1, 0)$. Hence $\rho(t) > \alpha_1$ in finite time t^*.

- $(\rho_0, d_0) \in \Omega_2 := \{(\rho, d) | 0 < \rho < \alpha_3,\ d < 0\}$.
  We will show that if $(\rho_0, d_0) \in \Omega_2$, then $(\rho(t), d(t)) \in \Omega_1$ in finite time. Suppose not, i.e., suppose $\rho(t) < \alpha_3$ for all $t > 0$. Then from Inequality (2.1a),

$$d' < -\frac{1}{2}d^2 - A(\rho-\alpha_1)(\rho-\alpha_2) < -\frac{1}{2}d^2 - M, \quad \text{for all } t > 0,$$

where $M := A(\alpha_3-\alpha_1)(\alpha_3-\alpha_2) > 0$. That is, $d' < -M$, for all $t > 0$ which, upon integrating over $[0,t]$, gives $d \le d_0 - Mt$. This tells us that

$$-\int_0^t d(s)ds \ge -d_0 t + \frac{Mt^2}{2}$$

and hence

$$\rho(t) = \rho_0 \exp\left(-\int_0^t d(s)ds\right) \ge \rho_0 \exp\left(-d_0 t + \frac{Mt^2}{2}\right).$$

But, since $d_0 < 0$ and $M > 0$, we will have $\rho(t) \ge \alpha_3$ in finite time which gives a contradiction.

- $(\rho_0,d_0)\in\Omega_3:=\{(\rho,d)|0<\rho\le\alpha_3 \text{ and } d\ge 0\}$.

  As long as $(\rho(t),d(t))\in\Omega_3$, ρ is decreasing and $d'<-\frac{1}{2}d^2-M\le -M$ where $M:=A(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)$. Therefore, $(\rho_0,d_0)\in\Omega_3$ implies $(\rho(t),d(t))\in\Omega_2$ in finite time.

- $(\rho_0,d_0)\in\Omega_4:=\left\{(\rho,d)\,\middle|\, \rho>\alpha_1,\ 0\le d<\sqrt{-2\rho\int_{\alpha_1}^{\rho}\frac{-As^2+k(s-1)}{s^2}\,ds}\right\}$.

  First, we show no orbit of the ODI touches the upper right branch of the separatirx curve; note that in Ω_4, since $\rho,d>0$, we have $\rho'\le 0$. Thus Inequality (2.8) leads to

$$\frac{d^2}{\rho}-\frac{d_0^2}{\rho_0}\le -2\int_{\rho_0}^{\rho}\frac{-As^2+k(s-1)}{s^2}\,ds. \tag{2.10}$$

Now, consider a point (ρ_0,d^*) on the invariant, i.e.,

$$\frac{{d^*}^2}{\rho_0}=-2\int_{\alpha_1}^{\rho_0}\frac{-As^2+k(s-1)}{s^2}\,ds.$$

Since $0<d_0<d_*$ we have

$$\begin{aligned}\frac{d^2}{\rho}&\le\frac{d_0^2}{\rho_0}-2\int_{\rho_0}^{\rho}\frac{-As^2+k(s-1)}{s^2}\,ds\\ &<\frac{d_*^2}{\rho_0}-2\int_{\rho_0}^{\rho}\frac{-As^2+k(s-1)}{s^2}\,ds\\ &=-2\int_{\alpha_1}^{\rho}\frac{-As^2+k(s-1)}{s^2}\,ds\end{aligned}$$

Hence, as long as $(\rho_0,d_0)\in\Omega_4$, no orbit of the ODI touches the upper right branch of the separatirx curve.

Next, we show that if $(\rho_0,d_0)\in\Omega_4$ then $(\rho(t),d(t))\nrightarrow(\alpha_1,0)$ as $t\to\infty$. Suppose $\rho(t)\searrow\alpha_1$ and $d(t)\searrow 0$ as $t\to\infty$. Then as $t\to\infty$, Inequality (2.10) leads to

$$-\frac{d_0^2}{\rho_0}\le -2\int_{\rho_0}^{\alpha_1}\frac{-As^2+k(s-1)}{s^2}\,ds.$$

But this contradicts the fact that $(\rho_0,d_0)\in\Omega_1$.

Finally, due to the non-touching result and the fact that $\lim_{t\to\infty}(\rho,d)\neq(\alpha_1,0)$, any orbit starting from within Ω_4 must enter $\Omega_5:=\{(\rho,d)|\rho>\alpha_1 \text{ and } d<0\}$ through $d=0$ and $\rho>\alpha_1$. Assume T^* is such a crossing time. Then

$$d'\le -A(\rho(T^*)-\alpha_1)(\rho(T^*)-\alpha_2),$$

which, upon integrating over $[0,T^*]$, gives

$$T^*\le\frac{d_0}{A(\rho(T^*)-\alpha_1)(\rho(T^*)-\alpha_2)}$$

which implies that T^* must be finite.

To sum up, we arrive at the following observation.

LEMMA 2.1. *If* $(\rho_0, d_0) \in \bigcup_{i=1}^4 \Omega_i$, *then* $(\rho(t), d(t)) \in \Omega_5$ *in finite time, where*

$$\Omega_5 := \{(\rho, d) |\ \alpha_1 < \rho \text{ and } d < 0\}.$$

Now in Ω_5, we shall carry out the blow-up analysis of

$$\begin{cases} d' \leq -\frac{1}{2}d^2 - A\rho^2 + k(\rho - c_b) = -\frac{1}{2}d^2 - A(\rho - \alpha_1)(\rho - \alpha_2) \\ \rho' = -d\rho \end{cases} \tag{2.11}$$

by a comparison to the corresponding ODE system

$$\begin{cases} e' = -\frac{1}{2}e^2 - A\zeta^2 + k(\zeta - c_b) = -\frac{1}{2}e^2 - A(\zeta - \alpha_1)(\zeta - \alpha_2) \\ \zeta' = -e\zeta. \end{cases} \tag{2.12}$$

The following lemma shows the monotonicity relation between the ODE and the ODI in Ω_5.

LEMMA 2.2. $\begin{cases} d(0) < e(0) < 0 \\ \zeta(0) < \rho(0), \end{cases}$ *implies* $\begin{cases} d(t) < e(t) < 0 \\ \zeta(t) < \rho(t), \end{cases}$ *for all* $t \geq 0$, *as long as* $\zeta(t) > \alpha_1$, *for all* $t \geq 0$.

Proof. We prove by way of contradiction. Suppose t_1 is the earliest time when the above assertion is violated. Then

$$\zeta(t_1) = \zeta(0) e^{-\int_0^{t_1} e(t)dt} < \rho(0) e^{-\int_0^{t_1} d(t)dt} = \rho(t_1).$$

Therefore, we are left with only one possibility: $e(t_1) = d(t_1)$.

From the original system (2.11) and the modified system (2.12),

$$(e - d)' \geq -\frac{1}{2}(e^2 - d^2) - A\{(\zeta - \alpha_1)(\zeta - \alpha_2) - (\rho - \alpha_1)(\rho - \alpha_2)\}. \tag{2.13}$$

Since $e(t) - d(t) > 0$ for $t < t_1$ and $e(t_1) - d(t_1) = 0$, at $t = t_1$ we have

$$(e(t_1) - d(t_1))' \leq 0.$$

But, since $\rho(t_1) > \zeta(t_1)$, when Inequality (2.13) is evaluated at $t = t_1$, it gives

$$-A\{(\zeta(t_1) - \alpha_1)(\zeta(t_1) - \alpha_2) - (\rho(t_1) - \alpha_1)(\rho(t_1) - \alpha_2)\} > 0.$$

This leads to a contradiction, as needed. □

The following lemma provides the blow-up conditions of the modified system (2.12), which in turn will lead to the blow-up of the original system (2.11).

LEMMA 2.3. *Consider the modified system* (2.12), *equipped with initial data* (ζ_0, e_0). *If* $(\zeta_0, e_0) \in \Omega_5$, *then* $e \to -\infty$, $\zeta \to \infty$ *in finite time.*

Proof. Consider

$$\begin{cases} e' = -\frac{1}{2}e^2 - A(\zeta - \alpha_1)(\zeta - \alpha_2), \\ \zeta' = -e\zeta. \end{cases}$$

Note that if $(\zeta_0,e_0)\in\Omega_5$, then $\zeta(t)$ is increasing for all t. Thus, $\zeta(t)>\alpha_1$, for all t. This implies $e'<-\frac{1}{2}e^2$ which, upon integration, yields

$$e(t)<\frac{2e_0}{te_0+2}.$$

This implies that

$$e(t)\to-\infty \quad\text{and}\quad \zeta(t)=\zeta_0\exp\left(-\int_0^t e(s)ds\right)\to\infty \quad\text{as } t\to t^*$$

with the blow-up time $t^*<-\frac{2}{e_0}$. □

Now we are ready for the last step of proving Theorem 1.1. We combine the monotonicity relation in Lemma 2.2 with Lemma 2.1 and Lemma 2.3. Consider any initial data $(\rho_0,d_0)\in\Omega_5$ for the ODI. Since Ω_5 is an open set, we can find $\epsilon>0$ such that $(\rho_0-\epsilon,d_0+\epsilon)\in\Omega_5$. We set this latter data as the initial data of the ODE for comparison purposes. This latter initial data will lead to finite time blow up of the ODE and thus the initial data $(\rho_0,d_0)\in\Omega_5$ will lead to finite time blow up of the ODI. Furthermore, by Lemma 2.2, we know that if an initial data of the ODI is contained in $\Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4$, then $(\rho(t),d(t))\in\Omega_5$ in finite time. Hence, initial data $(\rho_0,d_0)\in\Omega_1\cup\Omega_2\cup\Omega_3\cup\Omega_4\cup\Omega_5$ will lead to finite time blow-up of the original ODI.

To sum up, the above arguments give us the upper thresholds that lead to finite-time breakdown of the WREP equation. The threshold curve can be expressed as the union of two sets: One set is the half straight line

$$\{(\rho,d)\mid\rho=\alpha_3,\ d>0\},$$

and the other is a union of the lower-left arc and upper-right branches of the separatrix curve $\frac{d^2}{\rho}=-2\int_{\alpha_1}^{\rho}\frac{-As^2+k(s-1)}{s^2}ds$, i.e.,

$$\left\{(\rho,d)\;\middle|\;\rho\ge\alpha_3,\ d=\operatorname{sign}(\rho-\alpha_1)\sqrt{-2\rho\int_{\alpha_1}^{\rho}\frac{-As^2+k(s-1)}{s^2}ds}\right\}.$$

We complete the proof of Theorem 1.1 by recovering the normalized constant c_b, and expanding the above integral using the identities (2.7) and $B=2A\rho^2$.

3. Proof of Theorem 1.2

In this section we prove the existence of a one-sided threshold condition which leads to finite-time breakdown of the 2D WREP with attractive forcing ($k<0$). We shall carry out the blow-up analysis of

$$\begin{aligned} d' &\le -\frac{1}{2}d^2+\frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2-\left(\frac{\eta_0}{\rho_0}\right)^2\right\}\rho^2+k(\rho-c_b),\\ \rho' &= -d\rho, \end{aligned} \tag{3.1}$$

by comparing it to the corresponding ODE system

$$\begin{aligned} e' &= -\frac{1}{2}e^2+\frac{1}{2}\left\{\left(\frac{\omega_0}{\rho_0}\right)^2-\left(\frac{\eta_0}{\rho_0}\right)^2\right\}\zeta^2+k(\zeta-c_b),\\ \zeta' &= -e\zeta. \end{aligned} \tag{3.2}$$

As we did before, let $A := -\frac{1}{2}\{(\frac{\omega_0}{\rho_0})^2 - (\frac{\eta_0}{\rho_0})^2\} > 0$, and, for simplicity, we set $c_b = 1$. The following lemma shows the monotonicity relation between Systems (3.1) and (3.2). The proof is similar to that in [5], so details are omitted.

LEMMA 3.1. $\begin{cases} d(0) < e(0) \\ 0 < \zeta(0) < \rho(0) \end{cases}$ *implies* $\begin{cases} d(t) < e(t) \\ 0 < \zeta(t) < \rho(t) \end{cases}$ *for* $t \geq 0$, *as long as all solution remain finite on time interval* $[0,t]$.

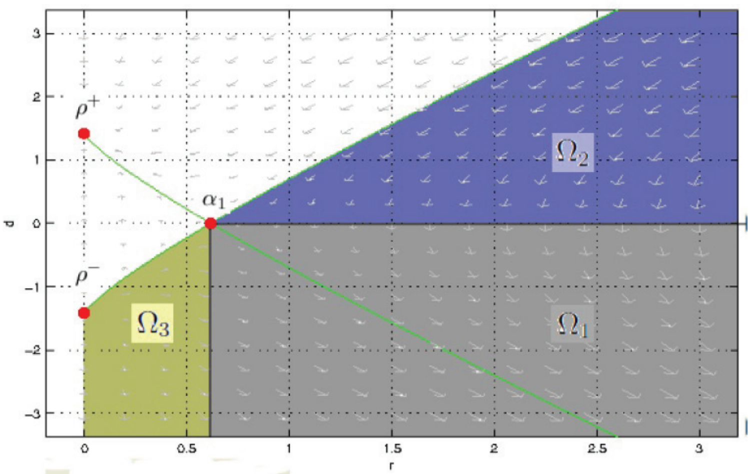

Fig. 3.1: *The blow up region of* $k < 0$ *case.*

System (3.2) admits three distinct critical points:

$$(\rho^{\pm}, d^{\pm}) := (0, \pm\sqrt{-2k}), \quad (\alpha_1, 0) := \left(\frac{k + \sqrt{k^2 - 4Ak}}{2A}, 0\right)$$

where (ρ^+, d^+) is a nodal sink, (ρ^-, d^-) is a nodal source, and $(\alpha_1, 0)$ is a saddle point. Also, as we did in the previous section, the separatrix curve passing through $(\alpha_1, 0)$ is given by

$$\frac{d^2}{\rho} = -2\int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\, ds.$$

Note that the comparison principle in Lemma 3.1 applies in $\Omega_1 := \{(\rho, d) | \rho > \alpha_1 \text{ and } d < 0\}$. We now discuss subcases distinguished by the location of initial points; see Figure 3.1.

- Case 1:

$$(\rho_0, d_0) \in \Omega_2 := \left\{(\rho, d) |\ \alpha_1 < \rho,\ 0 \leq d < \sqrt{-2\rho \int_{\alpha_1}^{\rho} \frac{-As^2 + k(s-1)}{s^2}\, ds}\right\}.$$

Due to the non-touching result we showed in the previous section, and the fact that $\lim_{t\to\infty}(\rho, d) \nrightarrow (\alpha_1, 0)$, we know that if $(\rho_0, d_0) \in \Omega_2$, then $(\rho(t), d(t)) \in \Omega_1$ in finite time. The proof of this is the same as in Ω_4 of the repulsive case.

- Case 2:

$$(\rho_0,d_0)\in\Omega_3:=\left\{(\rho,d)|\ 0<\rho\le\alpha_1,\ d<-\sqrt{-2\rho\int_{\alpha_1}^{\rho}\frac{-As^2+k(s-1)}{s^2}ds}\right\}.$$

As we did in the repulsive case, the non-touching result and the fact that $\lim_{t\to\infty}(\rho,d)\nrightarrow(\alpha_1,0)$ can be applied here too. We know that $\lim_{t\to\infty}d(t)\nrightarrow 0$. Thus $\rho(t)=\rho_0\exp(-\int_0^t d(s)ds)>\alpha_1$ in finite time.

To sum up, we arrive at the following observation.

Lemma 3.2. *If $(\rho_0,d_0)\in\Omega_2\cup\Omega_3$, then $(\rho(t),d(t))\in\Omega_1$ in finite time.*

The following lemma provides the blow-up conditions of the modified system in (3.2), which in turn, will lead to the blow-up of the original system in (3.1).

Lemma 3.3. *Consider the modified system* (3.2), *equipped with initial data (ζ_0,e_0). If $(\zeta_0,e_0)\in\Omega_1$, then $e\to-\infty$, $\zeta\to\infty$ in finite time.*

Proof. Consider

$$\begin{cases} e'=-\frac{1}{2}e^2-A(\zeta-\alpha_1)(\zeta-\beta),\\ \zeta'=-e\zeta.\end{cases}$$

where $\beta=\frac{k-\sqrt{k^2-4Ak}}{2A}<0$. Note that if $(\zeta_0,e_0)\in\Omega_1$, then $\zeta(t)$ is increasing in t. Thus, $\zeta(t)>\alpha_1$, for all t. This implies $e'<-\frac{1}{2}e^2$, which, upon integration, yields

$$e(t)<\frac{2e_0}{te_0+2}.$$

Hence, the blow-up time t^* of $e(t)$ must satisfy

$$t^*<-\frac{2}{e_0}.$$

Also, $e\to-\infty$ and $\zeta=\zeta_0\exp(-\int_0^t e(s)ds)\to\infty$. ◻

Now we are ready for the last step of proving Equation (1.2). We combine the monotonicity relation in Lemma 3.1 with Lemmas 3.2 and 3.3. Let us consider any initial data $(\rho_0,d_0)\in\Omega_1$ for the ODI. Since Ω_1 is an open set, we can find $\epsilon>0$ such that $(\rho_0-\epsilon,d_0+\epsilon)\in\Omega_1$. We set this latter data as the initial data of the ODE for comparison purposes. This latter initial data will lead to finite time blow up of the ODE and thus the initial data $(\rho_0,d_0)\in\Omega_1$ will lead to finite time blow up of the ODI. Furthermore, by Lemma 3.2, we know that if an initial data of the ODI is contained in $\Omega_2\cup\Omega_3$, then $(\rho(t),d(t))\in\Omega_1$ in finite time. Hence, initial data $(\rho_0,d_0)\in\Omega_1\cup\Omega_2\cup\Omega_3$ will lead to finite time blow up of the original ODI.

We close this section by stating the upper thresholds which determine the blow up region of the WREP equation. The upper right and lower left branches of

$$\frac{d^2}{\rho}=-2\int_{\alpha_1}^{\rho}\frac{-As^2+k(s-1)}{s^2}ds$$

are the critical thresholds. So the upper threshold can be expressed as

$$\left\{(\rho,d)\ |\rho>0,\ \ d=\mathrm{sign}(\rho-\alpha_1)\sqrt{-2\rho\int_{\alpha_1}^{\rho}\frac{-As^2+k(s-1)}{s^2}ds}\right\}.$$

Changing A back to $B=2A\rho^2$ we complete the proof of Theorem 1.2.

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